Supplementary file to
“Scaled Simplex Representation for Subspace Clustering”

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I. SOLUTION OF THE NLSR MODEL

The NLSR model (Eqn. (20) in the main paper) does not have an analytical solution. We employ a variable splitting method \cite{1}, \cite{2} to solve it. By introducing an auxiliary variable $Z$, we can reformulate the NLSR model into a linear equality-constraint problem with two variables $C$ and $Z$:

$$\min_{C, Z} \|X - XC\|_F^2 + \lambda\|C\|_F^2 \quad \text{s.t.} \quad Z = C, Z \geq 0. \quad (1)$$

Since the objective function is separable w.r.t. the variables $C$ and $Z$, problem (1) can be solved under the alternating direction method of multipliers (ADMM) \cite{3} framework. The Lagrangian function of the problem (1) is

$$L(C, Z, \Delta, \lambda, \rho) = \|X - XC\|_F^2 + \lambda\|C\|_F^2 + \langle \Delta, Z - C \rangle + \frac{\rho}{2}\|Z - C\|_F^2, \quad (2)$$

where $\Delta$ is the augmented Lagrangian multiplier and $\rho > 0$ is the penalty parameter. We initialize the variables $C_0$, $Z_0$, and $\Delta_0$ to be conformable zero matrices and set $\rho > 0$ with a suitable value. Denote by $(C_k, Z_k)$ and $\delta_k$ the optimization variables and the Lagrange multiplier at iteration $k$ ($k = 0, 1, 2, \ldots, K$), respectively. The variables can be updated by taking derivatives of the Lagrangian function (2) w.r.t. the variables $C$ and $Z$ and setting them to be zero.

1. Updating $C$ while fixing $Z$ and $\Delta$:

$$\min_{C} \|X - XC\|_F^2 + \lambda\|C\|_F^2 + \frac{\rho}{2}\|C - (Z_k + \rho^{-1}\Delta_k)\|_F^2. \quad (3)$$

This is a standard least squares regression problem with closed form solution:

$$C_{k+1} = (X^\top X + \frac{2\lambda + \rho}{2}I)^{-1}(X^\top X + \frac{\rho}{2}Z_k + \frac{1}{2}\Delta_k) \quad (4)$$

2. Updating $Z$ while fixing $C$ and $\Delta$:

$$\min_{Z} \|Z - (C_{k+1} - \rho^{-1}\Delta_k)\|_F^2 \quad \text{s.t.} \quad Z \geq 0. \quad (5)$$

The solution of $Z$ is

$$Z_{k+1} = \max(0, C_{k+1} - \rho^{-1}\Delta_k), \quad (6)$$

where the ”max(·)” operator outputs element-wisely the maximal value of the inputs.

3. Updating the Lagrangian multiplier $\Delta$:

$$\Delta_{k+1} = \Delta_k + \rho(Z_{k+1} - C_{k+1}). \quad (7)$$

The above alternative updating steps are repeated until the convergence condition is satisfied or the number of iterations exceeds a preset threshold $K$. The convergence condition of the ADMM algorithm is: $\|Z_{k+1} - C_{k+1}\|_F \leq \text{Tol}$, $\|C_{k+1} - C_k\|_F \leq \text{Tol}$, and $\|Z_{k+1} - Z_k\|_F \leq \text{Tol}$ are simultaneously satisfied, where $\text{Tol} > 0$ is a small tolerance value. Since the objective function and constraints are all strictly convex, the NLSR model solved by the ADMM algorithm \cite{3} is guaranteed to converge to a global optimal solution.

II. SOLUTION OF THE SLSR MODEL

We solve the SLSR model (Eqn. (21) in the main paper) by employing variable splitting methods \cite{1}, \cite{2}. Specifically, we introduce an auxiliary variable $Z$ into the SLSR model, which can then be equivalently reformulated as a linear equality-constrained problem:

$$\min_{C, Z} \|X - XC\|_F^2 + \lambda\|Z\|_F^2 \quad \text{s.t.} \quad 1^\top Z = s1^\top, Z = C, \quad (8)$$

whose solution for $C$ coincides with the solution of Eqn. (20) in the main paper. Since its objective function is separable w.r.t. the variables $C$ and $Z$, problem (8) can also be solved via the ADMM method \cite{3}. The corresponding augmented Lagrangian function is the same as in Eqn. (11) in the main paper. Denote by $(C_k, Z_k)$ and $\Delta_k$ the optimization variables and Lagrange multiplier at iteration $k$ ($k = 0, 1, 2, \ldots, K$), respectively. We initialize the variables $C_0$, $Z_0$, and $\Delta_0$ to be conformable zero matrices. By taking derivatives of the Lagrangian function $L$ (Eqn. (11) in the main paper) w.r.t. $C$ and $Z$, and setting them to be zeros, we can alternatively update the variables as follows:

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Algorithm 4: Projection of the vector \( v_{k+1} \) onto a scaled affine space

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Sort ( v_{k+1} ) into ( w: w_1 \geq w_2 \geq \ldots \geq w_N );</td>
</tr>
<tr>
<td>2.</td>
<td>Find ( \alpha = \max{1 \leq j \leq N : w_j + \frac{1}{\alpha}(s - \sum_{i=1}^N w_i) &gt; 0} );</td>
</tr>
<tr>
<td>3.</td>
<td>Define ( \beta = \frac{1}{\alpha}(s - \sum_{i=1}^N w_i) );</td>
</tr>
</tbody>
</table>

Output: \( z_{k+1} = v_{k+1} + \beta, i = 1, \ldots, N \).

(1) Updating \( C \) while fixing \( Z_k \) and \( \Delta_k \):

\[
C_{k+1} = \arg \min_C \| X - XC \|_F^2 + \frac{\rho}{2} \| C - (Z_k + \frac{1}{\rho} \Delta_k) \|_F^2. 
\]  

This is a standard least square regression problem and has a closed-from solution given by

\[
C_{k+1} = (X^\top X + \frac{\rho}{2} I)^{-1}(X^\top X + \frac{\rho}{2} Z_k + \frac{1}{2} \Delta_k). 
\]  

(2) Updating \( Z \) while fixing \( C_k \) and \( \Delta_k \):

\[
Z_{k+1} = \arg \min_Z \| Z - \frac{\rho}{2\lambda + \rho}(C_{k+1} - \rho^{-1} \Delta_k) \|_F^2 
\]  

s.t. \( 1^\top Z = s1^\top \).

This is a quadratic programming problem and the objective function is strictly convex, with a close and convex constraint, so there is a unique solution. Here, we employ the projection based method [4], whose computational complexity is \( O(N \log N) \) to process a vector of length \( N \). Denote by \( v_{k+1} \) an arbitrary column of \( \frac{\rho}{2\lambda + \rho}(C_{k+1} - \rho^{-1} \Delta_k) \), the solution of \( z_{k+1} \) (the corresponding column in \( Z_{k+1} \)) can be solved by projecting \( v_{k+1} \) onto a scaled affine space [4]. The solution of problem (11) is summarized in Algorithm 4.

(3) Updating \( \Delta \) while fixing \( C_k \) and \( Z_k \):

\[
\Delta_{k+1} = \Delta_k + \rho(Z_{k+1} - C_{k+1}). 
\]

We repeat the above alternative updates until a certain convergence condition is satisfied or the number of iterations reaches a preset threshold \( K \). The convergence condition of the ADMM algorithm is met when \( \| C_{k+1} - Z_{k+1} \|_F \leq \text{Tol} \), \( \| C_{k+1} - C_k \|_F \leq \text{Tol} \), and \( \| Z_{k+1} - Z_k \|_F \leq \text{Tol} \) are simultaneously satisfied, where \( \text{Tol} > 0 \) is a small tolerance value. Since the objective function and constraints are convex, the SLSR model solved by the ADMM algorithm, is guaranteed to converge to a global optimal solution.

REFERENCES


